

# CHARACTERIZING THE POWERSSET BY A COMPLETE (SCOTT) SENTENCE

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**ABSTRACT.** This paper is part II of a study on cardinals that are characterizable by a Scott sentence, continuing the work in [4]. A cardinal  $\kappa$  is characterized by a Scott sentence  $\phi_{\mathcal{M}}$ , if  $\phi_{\mathcal{M}}$  has a model of size  $\kappa$ , but no model of  $\kappa^+$ .

The main question in this paper is the following: Are the characterizable cardinals closed under the powerset operation? We prove that if  $\aleph_{\beta}$  is characterized by a Scott sentence, then  $2^{\aleph_{\beta+\beta_1}}$  is (homogeneously) characterized by a Scott sentence, for all  $0 < \beta_1 < \omega_1$  (cf. theorem 4.6 and definition 1.2). So, the answer to the above question is positive, except the case  $\beta_1 = 0$  which remains open.

As a consequence we derive that if  $\alpha \leq \beta$  and  $\aleph_{\beta}$  is characterized by a Scott sentence, then  $\aleph_{\alpha+\alpha_1}^{\aleph_{\beta+\beta_1}}$  is also characterized by a Scott sentence, for all  $\alpha_1 < \omega_1$  and  $0 < \beta_1 < \omega_1$  (cf. theorem 4.7). Whence, depending on the model of ZFC, we see that the class of characterizable and homogeneously characterizable cardinals is much richer than previously known. Several open questions are mentioned at the end.

## 1. INTRODUCTION

This paper is part II of a study on cardinals that are characterizable by a Scott sentence. We refer the reader to [4] for more details and background information. The main question we try to answer in this paper is the following: Are the characterizable cardinals closed under the powerset operation?

We prove a positive answer for all cardinals of the form  $\aleph_{\beta+\beta_1}$ , where  $\aleph_{\beta}$  is characterized by a Scott sentence and  $0 < \beta_1 < \omega_1$ . The case  $\beta_1 = 0$  remains open. The main construction is contained in theorem 4.1: Given a cardinal  $\aleph_{\beta}$  that is characterized by a Scott sentence, we prove that  $2^{\aleph_{\beta+1}}$  is also characterized by a Scott sentence. The idea is to create a complete graph whose edges are  $\aleph_{\beta+1}$ -colored and which “mimicks” the behavior of  $2^{\aleph_{\beta+1}}$  (cf. property 1). This will ensure that the graph can not have size greater than  $2^{\aleph_{\beta+1}}$  and the theorem follows.

Throughout the whole paper we work only with countable languages.

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**Basic Definitions.** We start by mentioning the basic definitions.

**Definition 1.1.** We say that a  $\mathcal{L}_{\omega_1, \omega}$ -sentence  $\phi$  *characterizes*  $\aleph_\alpha$ , or that  $\aleph_\alpha$  is *characterizable*, if  $\phi$  has models in all cardinalities up to  $\aleph_\alpha$ , but not in cardinality  $\aleph_{\alpha+1}$ . If  $\phi$  is the Scott sentence of a countable model (or any other complete sentence), we say that it *completely characterizes*  $\aleph_\alpha$ , or that  $\aleph_\alpha$  is *completely characterizable*. Denote by  $\mathcal{CH}_{\omega_1, \omega}$ , the set of all completely characterizable cardinals.

For now on, we will consider only completely characterizable cardinals, and we may just say characterizable cardinals.

**Definition 1.2.** If  $P$  is a unary predicate symbol, we say that it is *completely homogeneous* for the  $\mathcal{L}$ -structure  $\mathcal{A}$ , if  $P^{\mathcal{A}} = \{a \mid \mathcal{A} \models P(a)\}$  is infinite and every permutation of it extends to an automorphism of  $\mathcal{A}$ .

If  $\kappa$  is a cardinal, we will say that  $\kappa$  is *homogeneously characterizable* by  $(\phi_\kappa, P_\kappa)$ , if  $\phi_\kappa$  is a complete  $\mathcal{L}_{\omega_1, \omega}$ -sentence and  $P_\kappa$  a unary predicate in the language of  $\phi_\kappa$  such that

- $\phi_\kappa$  doesn't have models of power  $> \kappa$ ,
- if  $\mathcal{M}$  is the (unique) countable model of  $\phi_\kappa$ , then  $P_\kappa$  is infinite and completely homogeneous for  $\mathcal{M}$  and
- there is a model  $\mathcal{A}$  of  $\phi_\kappa$  such that  $P_\kappa^{\mathcal{A}}$  has cardinality  $\kappa$ .

If  $(\phi_\kappa, P_\kappa)$  characterize  $\kappa$  homogeneously, write  $(\mathcal{M}, P(\mathcal{M})) \models (\phi_\kappa, P_\kappa)$  for that. Denote the set of all homogeneously characterizable cardinals by  $\mathcal{HCH}_{\omega_1, \omega}$ . Obviously,  $\mathcal{HCH}_{\omega_1, \omega} \subset \mathcal{CH}_{\omega_1, \omega}$ , but the inverse inclusion fails.

Hjorth proved that the class of characterizable cardinals is closed under successors and countable unions, i.e. if  $\aleph_\alpha \in \mathcal{CH}_{\omega_1, \omega}$  and  $\beta < \omega_1$ , then  $\aleph_{\alpha+\beta} \in \mathcal{CH}_{\omega_1, \omega}$ . This means that characterizable cardinals come into clusters of length  $\omega_1$ .

**Definition 1.3.** A cardinal  $\aleph_\alpha \in \mathcal{CH}_{\omega_1, \omega}$  is called the *head of a cluster*, if we can not find ordinals  $\beta, \gamma$  such that

- $\aleph_\gamma \in \mathcal{CH}_{\omega_1, \omega}$ ,
- $\beta < \omega_1$  and
- $\aleph_\alpha = \aleph_{\gamma+\beta}$

It is immediate that all characterizable cardinals are of the form  $\aleph_{\alpha+\beta}$ , where  $\aleph_\alpha$  is the head of a cluster and  $\beta < \omega_1$ .

## 2. FRAISSE CONSTRUCTION

We describe here briefly a Fraisse-type construction which we will use in section 4. The following definitions and theorems are from [5], and the interested reader should refer there for more details<sup>1</sup>.

**Definition 2.1.** Let  $\mathcal{A}$  be a structure that contains  $\mathcal{M}$  and if  $A_0 \subset \mathcal{A}$ , then let  $\langle A_0 \rangle$  be the substructure of  $\mathcal{A}$  that is generated by  $A_0$ . We call finitely generated over  $\mathcal{M}$  the substructures of  $\mathcal{A}$  that have the form  $\langle A_0 \rangle \cup \mathcal{M}$ , where  $A_0$  is a finite subset of  $\mathcal{A} \setminus \mathcal{M}$ . We write finitely generated/ $\mathcal{M}$ .

If  $B_0 = \langle A_0 \rangle \cup \mathcal{M}$ ,  $B_1 = \langle A_1 \rangle \cup \mathcal{M}$  are finitely generated/ $\mathcal{M}$  substructures of  $\mathcal{A}$ , we write  $B_0 \subset B_1$  and we say that  $B_0$  is a *substructure* of  $B_1$ , if the same is true (in the usual sense) for  $\langle A_0 \rangle$  and  $\langle A_1 \rangle$ . We also write  $B_0 \cong B_1$  if there exists an isomorphism  $i : B_0 \rightarrow B_1$  such that  $i|_{\mathcal{M}} = id_{\mathcal{M}}$ .

It is straightforward to extend the above definition in the case were we have finitely many  $\mathcal{M}_0, \dots, \mathcal{M}_n$ .

Fraisse's theorem hold even for “finitely generated/ $\mathcal{M}$ ” substructures (For a proof of Fraisse's theorem one can consult [3]).

**Theorem 2.2.** (Fraisse) *Fix a countable model  $\mathcal{M}$  and let  $K(\mathcal{M})$  be a countable collection of finitely generated/ $\mathcal{M}$  structures (up to isomorphism). If  $K(\mathcal{M})$  has the Hereditary Property (HP), the Joint Embedding Property (JEP) and the Amalgamation Property (AP), then there is a countable structure  $\mathcal{F}$  which we will call the Fraisse limit of  $K(\mathcal{M})$ , such that*

- (1)  $\mathcal{F}$  is unique up to isomorphism and contains  $\mathcal{M}$ ,
- (2)  $K(\mathcal{M})$  is the collection of all finitely generated/ $\mathcal{M}$  substructures of  $\mathcal{F}$  (up to isomorphism), and
- (3) every isomorphism between finitely generated/ $\mathcal{M}$  substructures of  $\mathcal{F}$  extends to an automorphism of  $\mathcal{F}$ .

*The converse is also true, i.e. if  $\mathcal{F}$  is a countable structure such that every isomorphism between finitely generated/ $\mathcal{M}$  substructures of  $\mathcal{F}$  extends to an automorphism of  $\mathcal{F}$ , and  $K(\mathcal{M})$  is the collection of all finitely generated/ $\mathcal{M}$  substructures of  $\mathcal{F}$ , then  $K(\mathcal{M})$  has the HP, the JEP and the AP.*

**Theorem 2.3.** (Fraisse) *Fix a model  $\mathcal{M}$ . Assume that  $\mathcal{A}, \mathcal{B}$  are two structures (not necessarily countable) that contain  $\mathcal{M}$  and such that*

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<sup>1</sup>As mentioned in [5] too, these theorems are straightforward generalizations of Fraisse's theorems, and many authors have been using them implicitly, without writing them down explicitly.

- for every finitely generated/ $\mathcal{M}$  substructures  $C \subset D$  of  $\mathcal{A}$  (or of  $\mathcal{B}$ ), and every embedding  $f : C \hookrightarrow \mathcal{A}$  ( $f : C \hookrightarrow \mathcal{B}$ ), there is an embedding  $g : D \hookrightarrow \mathcal{A}$  ( $g : D \hookrightarrow \mathcal{B}$ ) that extends  $f$ , and
- the collection of all finitely generated/ $\mathcal{M}$  substructures of  $\mathcal{A}$  is the same as the collection of all finitely generated/ $\mathcal{M}$  substructures of  $\mathcal{B}$ .

Then  $\mathcal{A}$  and  $\mathcal{B}$  are back-and-forth equivalent, equivalently  $\mathcal{A} \equiv_{\infty, \omega} \mathcal{B}$ .

We now give a slightly different version of theorem 2.2 that will be more fitting to work with in the next section.

**Theorem 2.4.** *Fix a countable model  $\mathcal{M}$ . If  $K(\mathcal{M})$  is a countable collection of finitely generated/ $\mathcal{M}$  structures (up to isomorphism) and  $K(\mathcal{M})$  has the HP, the JEP and the AP, then there is a unique (up to isomorphism) countable structure  $\mathcal{F}$  that contains  $\mathcal{M}$  and satisfies the conjunction of*

- (I): *Every finitely generated/ $\mathcal{M}$  substructure of  $\mathcal{F}$  is in  $K(\mathcal{M})$ .*
- (II): *For every  $A_0$  finitely generated/ $\mathcal{M}$  substructure of  $\mathcal{F}$ , if  $A_1 \supset A_0$  and  $A_1 \in K(\mathcal{M})$ , then there exists some finitely generated/ $\mathcal{M}$  substructure  $B \subset \mathcal{F}$  and an isomorphism  $i : B \cong A_1$ , such that  $A_0 \subset B$  and  $i|_{A_0} = \text{id}$ .*

Moreover, if there is some  $\mathcal{L}_{\omega_1, \omega}$  sentence  $\psi$  such that  $A \in K(\mathcal{M})$  iff  $A \models \psi$  (as it will be the case in our example), then the conjunction of (I) and (II) can be written as a  $\mathcal{L}_{\omega_1, \omega}$ -sentence which is equivalent to the Scott sentence of  $\mathcal{F}$  and hence, it is complete.

**Corollary 2.5.** *If  $\mathcal{M}$  is countable and  $\mathcal{M}' \cong \mathcal{M}$ , then  $\lim K(\mathcal{M}') \cong \lim K(\mathcal{M})$ .*

### Notation:

- (1) In case we want to indicate which class we are talking about, we will write  $(I)_{K(\mathcal{M})}$  and  $(II)_{K(\mathcal{M})}$ .
- (2) If  $\phi$  is a complete sentence,  $\lim K(\phi)$  will denote  $\lim K(\mathcal{M})$ , where  $\mathcal{M}$  is some (any) countable  $\mathcal{M}$  that satisfies  $\phi$ . By corollary 2.5,  $\lim K(\phi)$  is uniquely determined (up to isomorphism).

Theorem 2.4 can be extended even in the case which  $\mathcal{M}$  and  $K(\mathcal{M})$  have cardinality  $\kappa > \aleph_0$ . The existence of  $\mathcal{F}$  in this case follows from the same diagonal argument as in the countable case, but the uniqueness of the Fraisse limit fails. However, all models of  $(I)_{K(\mathcal{M})}$  and  $(II)_{K(\mathcal{M})}$  will be  $\equiv_{\omega_1, \omega}$ -equivalent to each other (by theorem 2.3). So, we get the following

**Theorem 2.6.** *Let  $\psi$  be an  $\mathcal{L}_{\omega_1, \omega}$  sentence. Assume that  $\mathcal{M}$  is a countable model with Scott sentence  $\phi$  and  $\mathcal{N}$  is a model of  $\phi$  (possibly uncountable) and let  $K(\mathcal{M})$  be the collection of all finitely generated/ $\mathcal{M}$  substructures that satisfy  $\psi$  and let  $K(\mathcal{N})$  be the collection of all finitely generated/ $\mathcal{N}$  substructures that also satisfy  $\psi$ . Moreover, assume that  $K(\mathcal{M})$  and  $K(\mathcal{N})$  both have the HP, the JEP and the AP. Then any model of  $(I)_{K(\mathcal{N})}$  and  $(II)_{K(\mathcal{N})}$  is  $\equiv_{\infty, \omega}$ -equivalent to  $\lim K(\mathcal{M})$ .*

This proves that any Fraisse limit of  $K(\mathcal{N})$  satisfies the Scott sentence of  $\lim K(\mathcal{M})$ . If  $\mathcal{M}$  is a countable model whose Scott sentence  $\phi$  characterizes a certain cardinal  $\kappa$ , we will use the Scott sentence of  $\lim K(\mathcal{M})$  to characterize some cardinal  $\lambda \geq \kappa$ . In order to construct a Fraisse limit of  $K(\mathcal{N})$  we will use

**Theorem 2.7.** *Assume that  $\mathcal{M}$  is a countable model whose Scott sentence  $\phi$  characterizes an infinite cardinal  $\kappa$ ,  $\mathcal{N}$  is a model of  $\phi$  of cardinality  $\leq \kappa$ ,  $K(\mathcal{M})$  and  $K(\mathcal{N})$  are as above and  $\lambda \geq \kappa$ . Moreover, assume that:*

- (1) *If  $A$  is a finitely generated/ $\mathcal{N}$  structure, then there are  $\leq \lambda$  many (non-isomorphic) structures in  $K(\mathcal{N})$  that extend  $A$ , and*
- (2) *If  $\mathcal{G}$  is a structure such that*

$$\mathcal{N} \subset \mathcal{G}, |\mathcal{G} \setminus \mathcal{N}| \leq \lambda, \mathcal{G} \text{ satisfies } (I)_{K(\mathcal{N})}$$

*and for any  $A_0, A_1$  finitely generated/ $\mathcal{N}$  structures with*

$$A_0 \subset \mathcal{G}, A_1 \supset A_0 \text{ and } A_1 \in K(\mathcal{N}),$$

*then there is another structure  $\mathcal{G}'$  that extends  $\mathcal{G}$  and*

$$|\mathcal{G}' \setminus \mathcal{N}| \leq \lambda, \mathcal{G}' \text{ satisfies } (I)_{K(\mathcal{N})}$$

*and there is some finitely generated/ $\mathcal{N}$  structure  $B \subset \mathcal{G}'$  and an isomorphism  $i : B \cong A_1$ , with  $A_0 \subset B$  and  $i|_{A_0} = \text{id}$ .*

*Under the assumptions 1 and 2, we conclude that there is a structure  $\mathcal{G}^*$  with  $\mathcal{N} \subset \mathcal{G}^*$ ,  $|\mathcal{G}^*| = \lambda$  and  $\mathcal{G}^*$  satisfies  $(I)_{K(\mathcal{N})}$  and  $(II)_{K(\mathcal{N})}$ . Then  $\mathcal{G}^*$  also satisfies the Scott sentence of  $\lim K(\mathcal{M})$ .*

### 3. KNOWN THEOREMS

This section contains certain known theorems about characterizable cardinals. They are quoted from [1], [2], [4] and [5].

**Theorem 3.1** (theorem 2.9 from [4]). *If  $\kappa \in \mathcal{CH}_{\omega_1, \omega}$ , then one of the following is the case:*

- (1)  $\kappa^+ \in \mathcal{HCH}_{\omega_1, \omega}$  or,

(2)  $\kappa \in \mathcal{HCH}_{\omega_1, \omega}$ .

**Theorem 3.2** (theorem 3.4 from [4]). *If  $\lambda \in \mathcal{CH}_{\omega_1, \omega}$ , then  $\lambda^\omega \in \mathcal{HCH}_{\omega_1, \omega}$ .*

**Corollary 3.3** (corollary 3.6 from [4]). *If  $\kappa$  is an infinite cardinal and  $\lambda^\kappa \in \mathcal{CH}_{\omega_1, \omega}$ , then  $\lambda^\kappa \in \mathcal{HCH}_{\omega_1, \omega}$ .*

**Theorem 3.4** (theorem 3.7 from [4]). *If  $\aleph_\alpha^{\aleph_\beta} \in \mathcal{CH}_{\omega_1, \omega}$ , then for all  $\gamma < \omega_1$ ,*

$$\aleph_{\alpha+\gamma}^{\aleph_\beta} \in \mathcal{HCH}_{\omega_1, \omega}.$$

From [1] we have the following theorem, which also appears in [4]

**Theorem 3.5** (Baumgartner). *If  $\kappa \in \mathcal{HCH}_{\omega_1, \omega}$ , then  $2^\kappa \in \mathcal{HCH}_{\omega_1, \omega}$ .*

The following two theorems can be derived from [2] and appear in [4] and [5]<sup>2</sup> respectively.

**Theorem 3.6** (Hjorth). *Whenever  $\aleph_{\alpha_n}$ ,  $n \in \omega$ , is an non-decreasing sequence of cardinals in  $\mathcal{CH}_{\omega_1, \omega}$ , then  $\aleph_\lambda = \sup \aleph_{\alpha_n}$  is also in  $\mathcal{CH}_{\omega_1, \omega}$ .*

**Theorem 3.7.** (Hjorth) *If  $\kappa \in \mathcal{CH}_{\omega_1, \omega}$ , then at least one of the following holds:*

- (1)  $\kappa^+ \in \mathcal{HCH}_{\omega_1, \omega}$  or,
- (2) *there is a countable model  $\mathcal{M}$  in a language that contains a unary predicate  $P$  and a binary predicate  $<$  and whose Scott sentence  $\phi_{\mathcal{M}}$* 
  - (a) *has no models of cardinality  $\kappa^{++}$ ,*
  - (b) *does have a model of cardinality  $\kappa^+$  and*
  - (c) *for every model  $\mathcal{N}$  of  $\phi_{\mathcal{M}}$  of size  $\kappa^+$ ,  $(P^{\mathcal{N}}, <^{\mathcal{N}})$  is a dense linear ordering without endpoints and every initial segment of this linear ordering has size  $\kappa$ .*

With all these in place we are ready to prove some new theorems.

#### 4. POWERSSETS

**Theorem 4.1.** *If  $\kappa \in \mathcal{CH}_{\omega_1, \omega}$ , then  $2^{(\kappa^+)} \in \mathcal{HCH}_{\omega_1, \omega}$ .*

*Proof.* By corollary 3.3, if  $2^{(\kappa^+)} \in \mathcal{CH}_{\omega_1, \omega}$ , then it is also in  $\mathcal{HCH}_{\omega_1, \omega}$ . So, we just have to prove that  $2^{(\kappa^+)} \in \mathcal{CH}_{\omega_1, \omega}$ .

By theorem 3.7 we have to consider two cases. If  $\kappa^+ \in \mathcal{HCH}_{\omega_1, \omega}$  the result is immediate from theorem 3.5. If this is not the case, then there must be a complete sentence  $\phi_0$  in a language that contains a unary predicate  $P$  and a binary predicate  $<$  such that if  $\mathcal{N} \models \phi_0$ , then

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<sup>2</sup>The formulation of theorem 3.7 as it appears here is slightly different than the one in [5].

- $|\mathcal{N}| \leq \kappa^+$ ,
- $(P^\mathcal{N}, <^\mathcal{N})$  is a dense linear ordering without endpoints that has size  $\leq \kappa^+$  and
- for every  $m \in P^\mathcal{N}$ , the initial segment  $\{m_1 | m_1 <^\mathcal{N} m\}$  has size  $\leq \kappa$ .

Moreover, there is a certain model  $\mathcal{N}$  for which equality holds true in all the above properties. For such an  $\mathcal{N}$ , it holds that if  $\{m_\alpha | \alpha < \kappa\} \subset \mathcal{N}$  is an increasing  $<^\mathcal{N}$ -sequence, it can't be cofinal. We will use this later in the proof.

Now fix some countable  $\mathcal{M}$  that satisfies  $\phi_0$ . This  $\mathcal{M}$  is infinite and unique (up to isomorphism) and we will use the finitely generated/ $\mathcal{M}$  structures (cf. definition 2.1) for a Fraisse type construction. The idea behind the construction is to try to mimick the behavior of the powerset. If  $\kappa$  is an infinite cardinal and  $a, b \in 2^\kappa$ , then let  $f(a, b)$  be equal to the least  $\alpha \in \kappa$  such that  $a(\alpha) \neq b(\alpha)$ . It is immediate that all distinct  $a, b, c \in 2^\kappa$  satisfy the following property

**Property 1.** Exactly two of  $f(a, b)$ ,  $f(a, c)$  and  $f(b, c)$  are equal, while the third one is larger than the other two.

Since well-orders are not definable by  $\mathcal{L}_{\omega_1, \omega}$ , the above definition can not be given in  $\mathcal{L}_{\omega_1, \omega}$ . However, property 1 can be expressed in  $\mathcal{L}_{\omega_1, \omega}$  and is the one that drives the whole construction. We will consider all elements of  $2^\kappa$  as vertices of a complete graph and we will color all the edges between them using a function  $f$  that satisfies property 1.

Let  $V(\cdot), M(\cdot)$  be unary predicate symbols and  $f$  a binary function symbol. Let  $K$  be the collection of all countable structures  $\mathcal{A}$  that satisfy the conjunction of :

- (1)  $V(\mathcal{A}) \cup M(\mathcal{A})$  partition  $\mathcal{A}$ ,  $V(\mathcal{A})$  is finite and  $M(\mathcal{A}) = \mathcal{M}$ , where  $\mathcal{M}$  is as above. In particular,  $\mathcal{M} \models \phi_0$  and there is a linear order  $<^\mathcal{M}$  on  $\mathcal{M}$ . We will refer to  $M(\mathcal{A})$  as just  $\mathcal{M}$ .  $V(\mathcal{A})$  will be seen as a set of vertices in a complete graph, while  $\mathcal{M}$  will be an ordered set of colors.
- (2) If  $\Delta = \{(v, v) | v \in V(\mathcal{A})\}$ , then  $f : [V(\mathcal{A})]^2 \setminus \Delta \rightarrow \mathcal{M}$  and we think of  $f$  as assigning to every edge  $\{a, b\}$ ,  $a \neq b \in V(\mathcal{A})$ , a color in  $\mathcal{M}$ .
- (3) For every (distinct)  $a_0, a_1, a_2 \in V(\mathcal{A})$ , if  $f(\{a_0, a_1\}) \neq f(\{a_0, a_2\})$ , then

$$f(\{a_1, a_2\}) = \min\{f(\{a_0, a_1\}), f(\{a_0, a_2\})\}.$$

Otherwise,  $f(\{a_1, a_2\}) > f(\{a_0, a_1\}) = f(\{a_0, a_2\})$ .

Note that clause (3) is a reformulation of property 1. Also, note that (3) implies that if  $f(\{a, b\}) = m$ , then at least one of the  $f(\{a, c\}), f(\{b, c\})$  will get a value  $\leq m$ , for all  $a, b, c$ .

**Claim 1.**  $K$  satisfies J.E.P. and A.P.

*Proof.* For J.E.P., let  $\mathcal{A}, \mathcal{B}, \mathcal{C} \in K$ , with  $\mathcal{A} \subset \mathcal{B}, \mathcal{C}$ . We can assume that  $\mathcal{B} \cap \mathcal{C} = \mathcal{A}$  and we aim at making  $\mathcal{B} \cup \mathcal{C}$  to a structure in  $K$ . The problem is to properly define  $f$  on edges of the form  $\{b, c\}$ ,  $b \in \mathcal{B}$ ,  $c \in \mathcal{C}$ , in a way that (3) will be satisfied. Without loss of generality we can assume that

$$|V(\mathcal{B}) \setminus V(\mathcal{A})| = |V(\mathcal{C}) \setminus V(\mathcal{A})| = 1.$$

If they are not the case, we can just fix one  $b \in V(\mathcal{B}) \setminus V(\mathcal{A})$  and define by induction  $f(\{b, c\})$ , for all  $c \in V(\mathcal{C}) \setminus V(\mathcal{A})$ .

So, assume that  $V(\mathcal{B}) \setminus V(\mathcal{A}) = \{b\}$  and that  $V(\mathcal{C}) \setminus V(\mathcal{A}) = \{c\}$ . If there is  $a \in V(\mathcal{A})$  such that  $f(\{a, b\}) \neq f(\{a, c\})$ , then let  $f(\{b, c\}) = \min\{f(\{a, b\}), f(\{a, c\})\}$ . If this is not the case and for all  $a \in V(\mathcal{A})$   $f(\{a, b\}) = f(\{a, c\})$ , then choose  $m \in \mathcal{M}$  greater than all values used thus far (i.e. all the values in the image of  $f$ ) and assign  $f(\{b, c\}) = m$ . We now prove this is well-defined<sup>3</sup>:

Assume that there is some ambiguity, i.e. there are  $a_1 \neq a_2 \in V(\mathcal{A})$  such that  $f(\{a_i, b\}) \neq f(\{a_i, c\})$ , for both  $i = 1, 2$  and

$$m_1 = \min\{f(\{a_1, b\}), f(\{a_1, c\})\} < m_2 = \min\{f(\{a_2, b\}), f(\{a_2, c\})\}.$$

Assume that  $m_1 = f(\{a_1, b\})$  (work similarly if  $m_1 = f(\{a_1, c\})$ ). Then,  $a_1, a_2, b$  are all in  $V(\mathcal{B})$  and it holds that  $f(\{a_1, b\}) = m_1 < m_2 \leq f(\{a_2, b\})$ . By our assumption  $\mathcal{B} \in K$ , we conclude that

$$f(\{a_1, a_2\}) = \min\{f(\{a_1, b\}), f(\{a_2, b\})\} = f(\{a_1, b\}) = m_1.$$

By the note made immediately after the definition of  $K$ ,  $f(\{a_1, a_2\}) = m_1$  implies that one of the  $f(\{a_1, c\}), f(\{a_2, c\})$  has a value  $\leq m_1$ . But  $m_1 < m_2 \leq f(\{a_2, c\})$ . So, it has to be that  $f(\{a_1, c\}) \leq m_1$ . But, in this case,  $f(\{a_1, b\}) = f(\{a_1, c\}) = m_1$  which contradicts the assumption that  $f(\{a_1, b\}) \neq f(\{a_2, b\})$ .

Subsequently, it is not hard to see that  $f(\{b, c\})$  was defined in such a way that property (3) of the definition of  $K$  is preserved. The above proof gives us that for all  $a \in V(\mathcal{A})$  with  $f(\{a, b\}) \neq f(\{a, c\})$ ,  $f(\{b, c\}) = \min\{f(\{a, c\}), f(\{a, b\})\}$ . Therefore, (3) is preserved for any  $a \in V(\mathcal{A})$  and for  $b, c$ .

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<sup>3</sup>If the reader wants to verify the following claims, it is helpful to draw a diagram with all the points and vertices.



Further, assume that for  $a \in V(\mathcal{A})$  and  $f(\{a, b\}) = f(\{a, c\})$ . If  $f(\{b, c\})$  was given a value  $m$  greater than all values used thus far, then it also holds that  $f(\{b, c\}) > f(\{a, b\}) = f(\{a, c\})$ , which gives property (3). If this is not the case, then there exists some  $a_1 \neq a$  such that, say,  $f(\{a_1, b\}) < f(\{a_1, c\})$  and  $f(\{b, c\})$  was defined to be equal to  $\min\{f(\{a_1, b\}), f(\{a_1, c\})\} = f(\{a_1, b\})$ .

If  $f(\{a_1, b\}) = f(\{a, b\})$ , then  $f(\{a_1, b\}) = f(\{a, c\})$  and  $f(\{a_1, c\}) > f(\{a, c\})$ , which means that by  $\mathcal{C}$  being a structure in  $K$ ,  $f(\{a, a_1\}) = \min\{f(\{a_1, c\}), f(\{a, c\})\} = f(\{a, c\})$  and in the triangle formed by the vertices  $a, b, a_1$  all vertices have the same value, which contradicts (3) for  $\mathcal{B}$ .

If  $f(\{a_1, b\}) < f(\{a, b\})$ , then  $f(\{a, a_1\}) = \min\{f(\{a_1, b\}), f(\{a, b\})\} = f(\{a_1, b\})$ . But, as we noted, this implies that at least one of the  $f(\{a, c\})$ ,  $f(\{a_1, c\})$  gets value  $\leq f(\{a_1, b\})$ . Contradiction, since  $f(\{a, c\}) = f(\{a, b\}) > f(\{a_1, b\})$  and we also assumed that  $f(\{a_1, c\}) > f(\{a_1, b\})$ .

Therefore, it must be the case that  $f(\{b, c\}) = f(\{a_1, b\}) > f(\{a, b\}) = f(\{a, c\})$ , which is exactly what (3) says. This proves that by the way we defined  $f$ ,  $\mathcal{B} \cup \mathcal{C} \in K$ .

Finally, for A.P., let  $\mathcal{B}, \mathcal{C} \in K$ . We can assume that  $\mathcal{B} \cap \mathcal{C} = \mathcal{M}$  (equivalently  $V(\mathcal{B}) \cap V(\mathcal{C}) = \emptyset$ ). Fix  $b \in \mathcal{B}$  and  $c \in \mathcal{C}$  and assign  $f(\{b, c\})$  to be *any* value. Then, work as above.  $\square$

The Hereditary Property (H.P.) is immediate, and therefore, there exists a countable structure  $\mathcal{F}$  as in theorem 2.4. Let  $\phi_{\mathcal{F}}$  be its Scott sentence. By corollary 2.5, the choice of  $\mathcal{M}$  is not of importance, as different choices will yield isomorphic limits. In either case,  $M(\mathcal{F}) \cong \mathcal{M}$  and therefore,  $M(\mathcal{F}) \models \phi_0$ .

**Claim 2.** If  $\mathcal{G} \models \phi_{\mathcal{F}}$ , then  $|\mathcal{G}| \leq 2^{(\kappa^+)}$ .

*Proof.* Towards contradiction assume not and let  $\mathcal{G}$  witness this. Without loss of generality, assume that  $|\mathcal{G}| = (2^{\kappa^+})^+ = (\beth_1(\kappa^+))^+$ . Since  $M(\mathcal{G}) \models \phi_0$ , it must be  $|M(\mathcal{G})| \leq \kappa^+$ , which further implies that  $|V(\mathcal{G})| = (\beth_1(\kappa^+))^+$ . Hence,  $f^{\mathcal{G}}$  gives a function from  $[(\beth_1(\kappa^+))^+]^2$  to  $\kappa^+$ .

By Erdős-Rado theorem,

$$\beth_1(\kappa^+)^+ \rightarrow (\kappa^{++})_{\kappa^+}^2,$$

which also implies

$$\beth_1(\kappa^+)^+ \rightarrow (3)_{\kappa^+}^2.$$

Applying this to  $f^{\mathcal{G}}$  we get that there is a homogeneous set of size 3, i.e. there are  $a, b, c \in V(\mathcal{G})$  such that

$$f^{\mathcal{G}}(a, b) = f^{\mathcal{G}}(b, c) = f^{\mathcal{G}}(a, c),$$

which contradicts the way  $\mathcal{F}$  was defined and the fact that  $\mathcal{G} \models \phi_{\mathcal{F}}$ .  $\square$

**Claim 3.** There is  $\mathcal{G}^* \models \phi_{\mathcal{F}}$  of size  $2^{\kappa^+}$ .

*Proof.* We can assume here that  $M(\mathcal{G}^*)$  has size exactly  $\kappa^+$ . By theorem 2.7 we just need to prove that if  $\mathcal{G}$  is such that  $|G| < 2^{\kappa^+}$  and satisfies  $(I)_K$  (see theorem 2.4 for the definition of  $(I)_K$ ),  $\mathcal{A} \subset \mathcal{G}$ ,  $\mathcal{A} \subset \mathcal{B}$ ,  $\mathcal{A}, \mathcal{B} \in K$ , then, there is structure  $\mathcal{G}' \supset \mathcal{G}$  that also satisfies  $(I)_K$ , and there is  $\mathcal{C} \subset \mathcal{G}'$  and  $i : \mathcal{B} \cong \mathcal{C}$  with  $i|_{\mathcal{A}} = id$ .

As previously, we can assume that  $V(\mathcal{B}) \setminus V(\mathcal{A}) = \{b\}$  (if this is not the case, then proceed by induction on  $V(\mathcal{B}) \setminus V(\mathcal{A})$ ). Let  $\mathcal{G}' = \mathcal{G} \cup \{b\}$ . We keep  $f$  on  $\mathcal{G}$  and  $\mathcal{B}$  and we need to define  $f$  on edges of the form  $\{b, v\}$ , for all  $v \in V(\mathcal{G})$ , such that property (3) of the definition of  $K$  is satisfied. We need some preliminary work before that.

**Definition 4.2.** Fix  $\mathcal{G}$  as above. A sequence  $\vec{A} = \langle A_\alpha \rangle$  of length  $\kappa^+$  will be called a *selector*, if

- $A_0 \subset V(\mathcal{G})$ ,
- $A_\lambda = \bigcap_{\alpha < \lambda} A_\alpha$ , for  $\lambda$  limit ordinal, and
- $A_{\alpha+1} = \emptyset$ , if  $A_\alpha = \emptyset$ .
- Otherwise, there are  $v_\alpha, \lambda_\alpha$  such that  $v_\alpha \in A_\alpha$ ,  $\lambda_\alpha \in M(\mathcal{G})$ ,  $\lambda_\alpha > \lambda_\beta$ , for all  $\beta < \alpha$  and

$$A_{\alpha+1} = \{v \in A_\alpha \mid f(\{v, v_\alpha\}) = \lambda_\alpha\}.$$

In this case, write  $A_{\alpha+1} = Q(A_\alpha, v_\alpha, \lambda_\alpha)$  to indicate their dependence.

So, at every stage we choose an element  $v_\alpha$  of  $A_\alpha$  and an element  $\lambda_\alpha$  in  $M(\mathcal{G})$ , such that the  $\lambda_\alpha$ 's form a strictly increasing sequence, and we form the set  $A_{\alpha+1}$  as above. We repeat that until we either get the empty set, or we complete  $\kappa^+$  many steps.

The selector will be called *good*, if

$$\bigcap_{\alpha < \kappa^+} A_\alpha = \emptyset.$$

Otherwise, it will be called *bad*.

**Claim 4.** For all selectors  $|\bigcap_{\alpha < \kappa^+} A_\alpha| \leq 1$ .

*Proof.* This is obvious for good selectors, so let  $\vec{A}$  be a bad selector with  $a, b \in \bigcap_{\alpha < \kappa^+} A_\alpha \subset V(\mathcal{G})$  and  $f(\{a, b\}) = \lambda$ . It is immediate that for all  $\alpha$ ,  $A_\alpha \neq \emptyset$  and, therefore, at every stage some  $v_\alpha, \lambda_\alpha$  are chosen. This goes on for  $\kappa^+$  many stages and, eventually, we will get an increasing sequence of  $\lambda_\alpha$ 's

of length  $\kappa^+$ . This has to be cofinal, which means that we can choose  $\alpha < \kappa^+$  such that  $\lambda_\alpha > \lambda$ . Then,  $a, b \in \bigcap_{\alpha < \kappa^+} A_\alpha$  implies also that  $a, b \in A_{\alpha+1}$ , or that

$$f(\{a, v_\alpha\}) = f(\{b, v_\alpha\}) = \lambda_\alpha.$$

This contradicts property (3) for  $a, b, v_\alpha$ , since  $\lambda_\alpha > \lambda = f(\{a, b\})$ .

Hence,  $|\bigcap_{\alpha < \kappa^+} A_\alpha| = 1$ , for bad selectors.  $\square$

**Definition 4.3.** For a bad selector  $\vec{A}$ , call  $v(\vec{A})$  the unique element of its intersection.

For selectors  $\vec{A}, \vec{B}$ , let  $\vec{A} \sim \vec{B}$  if and only if

$$\bigcap_{\alpha < \kappa^+} A_\alpha = \bigcap_{\alpha < \kappa^+} B_\alpha.$$

For bad selectors, this is the case iff  $v(\vec{A}) = v(\vec{B})$ .

This defines an equivalence relation on the selectors, each one associated with a (unique) element in  $V(\mathcal{G})$ . So, it is immediate that there are exactly  $|V(\mathcal{G})| < 2^{\kappa^+}$  many equivalent classes. We will use that to prove that there is at least one good selector. We need the following claim before that:

**Claim 5.** If  $\vec{A}, \vec{B}$  are bad selectors such that for all  $\alpha \leq \alpha_0$ ,  $A_\alpha = B_\alpha \neq \emptyset$  and

$$A_{\alpha_0+1} = Q(A_{\alpha_0}, v_{\alpha_0}, \lambda_{\alpha_0}), \quad B_{\alpha_0+1} = Q(B_{\alpha_0}, v_{\alpha_0}, \lambda'_{\alpha_0}),$$

with  $\lambda_{\alpha_0} \neq \lambda'_{\alpha_0}$ , then  $\vec{A} \not\sim \vec{B}$ .

*Proof.* Note that  $v_{\alpha_0}$  is the same in both cases, while  $\lambda_{\alpha_0}$  is not. If  $v = v(\vec{A}) = v(\vec{B})$ , then we get on the one hand that  $f(\{v, v_{\alpha_0}\}) = \lambda_{\alpha_0}$  and on the other hand  $f(\{v, v_{\alpha_0}\}) = \lambda'_{\alpha_0}$ . Contradiction.  $\square$

**Claim 6.** For all  $W \subset V(\mathcal{G})$  there is a good selector  $\vec{A}$  with  $A_0 = W$ .

*Proof.* Let  $W \subset V(\mathcal{G})$ , non-empty, and assume that all selectors with  $A_0 = W$  are bad. Let  $H$  be the set of functions from  $\kappa^+$  to  $M(\mathcal{G})$  that are increasing. There are  $2^{\kappa^+}$  many of them. For every  $h \in H$  we will define a selector  $\vec{A}_h$ , in such a way that if  $h \neq h'$ , then  $\vec{A}_h \not\sim \vec{A}_{h'}$ . This will give us  $2^{\kappa^+}$  many bad selectors, which we saw is a contradiction.

Define  $A_{h,0} = W$ , for all  $h \in H$  and  $A_{h,\gamma} = \bigcap_{\alpha < \gamma} A_{h,\alpha}$  for  $\gamma$  limit. If  $A_{h,\alpha}$  is given, by the assumption that all selectors are bad, necessarily it must be non-empty. Choose  $v_{h,\alpha}$  in it and let

$$A_{h',\alpha+1} = Q(A_{h',\alpha}, v_{h,\alpha}, h'(\alpha)),$$

for all  $h' \in H$  with  $h'|_\alpha = h|_\alpha$ , or that all functions that agree with  $h$  up to  $\alpha$  are assigned the same element  $v_{h,\alpha}$ . If  $h'(\alpha) \neq h(\alpha)$ , then, by the previous

lemma,  $\vec{A}_{h'} \approx \vec{A}_h$ . Since all functions differ from each other at a point, we get that all the  $\vec{A}_h$  are non-equivalent. Contradiction.

Therefore, there must exist a good selector, call it  $\vec{A} = (A_\alpha)_{\alpha < \kappa^+}$  with  $A_0 = W$ .  $\square$

We will actually need slightly more than that. Assume that there is a predetermined value  $\mu$  and we are interested only in these  $h' \in H$  such that  $h'(\alpha) > \mu$ , for all  $\alpha < \kappa^+$ . Then, the same argument as above works. So, assume that for  $\vec{A}$ , a good selector as above, if  $A_{\alpha+1} = Q(A_\alpha, v_\alpha, \lambda_\alpha)$ , then  $\lambda_\alpha > \mu$ . Also, we observe that by  $\vec{A}$  being good, for all  $v \in A_0 = W$  there exists  $\alpha$  with  $v \in A_\alpha \setminus A_{\alpha+1}$ .

Back to  $b$  being the unique element in  $V(\mathcal{B}) \setminus V(\mathcal{A})$ , we need to define  $f(\{b, v\})$ , for all  $v \in V(\mathcal{G}) \setminus V(\mathcal{A})$ . Define sets  $V_n \subset V(\mathcal{G})$  as follows

- $V_0 = V(\mathcal{A})$ .
- $V_{n+1} = \{v \in V(\mathcal{G}) \mid \exists a \in V_n (f(\{a, b\}) \neq f(\{a, v\}))\}$ .

Let  $V_\omega = \cup_n V_n$  and if  $v \in V_{n+1}$ , define  $f(\{b, v\}) = \min\{f(\{a, v\}), f(\{a, b\})\}$ , where  $a$  is given by the fact that  $v \in V_{n+1}$ . By induction on  $n$ , repeating the argument that we gave for  $K$  satisfying J.E.P., we can prove that  $f(\{b, v\})$  defined above is well-defined and satisfies property (3) of the definition of  $K$ . Thus, it remains to define  $f(\{b, w\})$  for  $w \in W = V(\mathcal{G}) \setminus V_\omega$ .

We first observe that

$$W = \{w \in V(\mathcal{G}) \mid \forall a \in V_\omega f(\{a, b\}) = f(\{a, w\})\}.$$

In a way, every  $a \in V_\omega$  is imposing a restriction on  $w$  that once we define  $f(\{w, b\})$  it must be greater than  $f(\{b, a\}) = f(\{a, w\})$  (if we want (3) to be holding). On the face of it, we have many restrictions to worry about, but in fact there are only finitely many as seen by the following claim.

**Claim 7.** For every  $w \in W$ , if  $f(\{b, w\})$  gets defined to be  $> f(\{b, a\})$ , for  $a \in V(\mathcal{A}) = V_0$ , then it is  $> f(\{b, a\})$  for all  $a \in V_\omega$ .

*Proof.* By induction on  $n$ . Assume that this is true for  $V_n$  and that  $v \in V_{n+1}$  as witnessed by  $a \in V_n$  (i.e.  $f(\{a, b\}) \neq f(\{a, v\})$ ). Then,  $f(\{b, v\}) = \min\{f(\{a, b\}), f(\{a, v\})\} \leq f(\{a, b\})$ . So, if we define  $f(\{b, w\}) > f(\{a, b\})$ , then also  $f(\{b, w\}) > f(\{a, b\}) \geq f(\{b, v\})$ .  $\square$

The key point here is that these finitely many restrictions don't even depend on  $w$ . So, define

$$\mu = \max\{f(\{b, a\}) = f(\{a, w\}) \mid a \in V_0\}$$

and it suffices to define  $f(\{b, w\}) > \mu$ . Here is where we use the fact that there exists a good selector  $\vec{A}$ , for which  $A_0 = W$  and such that  $\lambda_\alpha > \mu$  for all  $\alpha$ .

Define

$$f(\{b, v_\alpha\}) = \lambda_\alpha, \text{ for all } \alpha$$

and if  $w \in A_\alpha \setminus A_{\alpha+1}$ , then let

$$f(\{b, w\}) = \min\{f(\{w, v_\alpha\}), f(\{b, v_\alpha\}) = \lambda_\alpha\}.$$

As we saw, for all  $w \in W$ , there exists such an  $\alpha$ , which means that  $f(\{b, w\})$  is defined for all  $w$ .

**Claim 8.**  $f(\{b, w\})$  as defined above satisfies property (3) of the definition of  $K$ .

*Proof.* We saw that this is true for elements in  $V_\omega$ . So, assume we have  $b$  and another two elements  $a_1, a_2$ . There are several case<sup>4</sup>:

- Both  $a_1, a_2 \in W$  and  $f(\{b, a_1\}), f(\{b, a_2\})$  were defined at the same stage, say  $\alpha$ . Let  $x_1 = f(\{v_\alpha, a_1\}), x_2 = f(\{v_\alpha, a_2\}), x_1, x_2 \neq \lambda_\alpha$ . Then,

$$f(\{b, a_1\}) = \min\{x_1, \lambda_\alpha\}, f(\{b, a_2\}) = \min\{x_2, \lambda_\alpha\} \text{ and}$$

$$f(\{a_1, a_2\}) = \min\{x_1, x_2\}, \text{ or } f(\{a_1, a_2\}) > x_1, \text{ if } x_1 = x_2.$$

It is not hard to see that given  $x_1 \neq x_2$ , a triangle with vertices  $\min\{x_1, \lambda_\alpha\}, \min\{x_2, \lambda_\alpha\}, \min\{x_1, x_2\}$  has two of its vertices being equal and the other one greater.

If  $x_1 = x_2$  and say  $x_1 < \lambda_\alpha$ , then we get

$$f(\{b, a_1\}) = f(\{b, a_2\}) = x_1 < f(\{a_1, a_2\}).$$

If  $x_1 = x_2$  and say  $x_1 > \lambda_\alpha$ , then

$$f(\{b, a_1\}) = \lambda_\alpha = f(\{b, a_2\}) < x_1 < f(\{a_1, a_2\}).$$

- Let  $a_1 = a \in W$ , with  $f(\{b, a\})$  having been defined at stage  $\beta$  and  $a_2 = v_\alpha$ , for some  $\alpha > \beta$ . Let  $f(\{a, v_\beta\}) = x$ . Since  $v_\alpha \in A_\alpha \supset A_{\beta+1}$ , it must be that  $f(\{v_\alpha, v_\beta\}) = \lambda_\beta$ . So, by property (3)

$$f(\{a, v_\alpha\}) = \min\{f(\{a, v_\beta\}), f(\{v_\beta, v_\alpha\})\} = \min\{x, \lambda_\beta\} = f(\{b, a\}).$$

Since

$$f(\{b, v_\alpha\}) = \lambda_\alpha > \lambda_\beta \geq \min\{x, \lambda_\beta\},$$

$a, b, v_\alpha$  form a triangle with two vertices having two equal value and the third one being greater.

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<sup>4</sup>Again, it helps to draw a diagram with all the points and vertices

- If  $a, \beta, v_\alpha$  are as above with the exception that now  $\beta > \alpha$ , we work similarly. Note that  $a, v_\beta \in A_\beta \supset A_{\alpha+1}$  implies that

$$f(\{b, v_\alpha\}) = \lambda_\alpha = f(\{v_\alpha, a\}) = f(\{v_\alpha, v_\beta\}).$$

Property (3) then applied on the triangle with vertices  $\{v_\alpha, v_\beta, a\}$  gives that  $\lambda_\alpha < f(\{v_\beta, a\})$ . This combined with the fact that  $\lambda_\alpha < \lambda_\beta$ , gives

$$f(\{b, v_\alpha\}) = f(\{v_\alpha, a\}) = \lambda_\alpha < \min\{f(\{v_\beta, a\}), \lambda_\beta\} = f(\{b, a\}).$$

- Let  $a_1, a_2 \in W$  with  $f(\{b, a_1\})$  being defined at stage  $\beta$ , while  $f(\{b, a_2\})$  being defined at stage  $\alpha > \beta$ . Also let  $f(\{a_1, v_\beta\}) = x \neq \lambda_\beta$  and  $f(\{a_2, v_\alpha\}) = y \neq \lambda_\alpha$ . We saw previously  $a_2 \in A_\alpha \supset A_{\beta+1}$  implies that  $f(\{a_2, v_\beta\}) = \lambda_\beta$ . Therefore,

$$f(\{a_1, a_2\}) = \min\{f(\{a_1, v_\beta\}), f(\{v_\beta, a_2\}) = \min\{x, \lambda_\beta\} = f(\{b, a_1\}).$$

It also holds that

$$f(\{b, a_2\}) = \min\{y, \lambda_\alpha\} > \lambda_\beta \geq \min\{x, \lambda_\beta\},$$

with the (strict) inequality by a similar calculation as in the previous case. Putting all these together, the vertices  $\{a_1, a_2\}, \{b, a_1\}$  get the same value, while  $\{b, a_2\}$  gets a greater value.

- Assume we have  $b, v_\alpha$  and  $a \in V_\omega$ . Since  $v_\alpha \in W$ , it must be that  $f(\{b, a\}) = f(\{a, v_\alpha\})$ . Since  $f(\{b, v_\alpha\}) = \lambda_\alpha$  and we assume that all  $\lambda_\alpha$  are greater than  $\mu = \max\{f(\{b, a'\}) | a' \in V_0\} = \max\{f(\{b, a''\}) | a'' \in V_\omega\}$ , we get

$$f(\{b, a\}) = f(\{a, v_\alpha\}) \leq \mu < \lambda_\alpha.$$

- Similarly, if  $a \in V_\omega$  and  $w \in W$ , with  $f(\{b, w\})$  being defined at stage  $\alpha$ , with similar calculations as before we get that  $f(\{b, a\}) = f(\{a, v_\alpha\}) = f(\{a, w\}) \leq \mu < \lambda_\alpha$ . Moreover,

$$f(\{w, v_\alpha\}) > f(\{a, w\}) = f(\{a, v_\alpha\}),$$

which further implies that

$$f(\{b, w\}) = \min\{f(\{w, v_\alpha\}), \lambda_\alpha\} > f(\{a, w\}) = f(\{b, a\}),$$

which gives property (3).

Thus, in all cases, property (3) is satisfied.  $\square$

This enables us to extend  $f$  from  $V(\mathcal{G})$  to  $V(\mathcal{G}) \cup \{b\}$  in such a way that all finitely generated substructures of  $\mathcal{G} \cup \{b\}$  are in  $K$ . For every finitely generated substructure there are  $2^{(\kappa^+)}$  many different structures in  $K$  that extend it. Applying theorem 2.7 we get  $\mathcal{G}^*$  of size  $2^{(\kappa^+)}$  that satisfies  $\phi_{\mathcal{F}}$ .  $\square$

This  $\mathcal{G}^*$  is a structure that witnesses that  $2^{(\kappa^+)} \in \mathcal{CH}_{\omega_1, \omega}$ .  $\square$

**Remark:** The only facts that we used in the above proof about  $\phi_0$  and the linear ordering  $<$  are the following:

- (1) There is **no** model of  $\phi_0$  where  $<$  has size  $\kappa^{++}$ .
- (2) There is a model of  $\phi_0$  with an increasing  $<$ -sequence of size  $\kappa^+$ .

This gives us the following:

**Theorem 4.4.** *Let  $\phi$  be a complete sentence such that*

- (1) *For every model  $\mathcal{M}$  of  $\phi$ ,  $<^{\mathcal{M}}$  is a linear order.*
- (2)  *$\phi$  does not have any model where  $<^{\mathcal{M}}$  has cardinality  $\lambda^+$ .*
- (3)  *$\phi$  has a model  $\mathcal{M}$  which has an increasing  $<^{\mathcal{M}}$ -sequence of size  $\lambda$ .*

*Then  $2^\lambda$  is characterizable.*

*Proof.* By the proof of theorem 4.1 and the above remark.  $\square$

**Theorem 4.5.** *If  $\aleph_\lambda = \sup_n \aleph_{\alpha_n}$  and  $\aleph_{\alpha_n} \in \mathcal{CH}_{\omega_1, \omega}$ , then  $2^{\aleph_\lambda}$  is in  $\mathcal{HCH}_{\omega_1, \omega}$ .*

*Proof.* By the theorem 4.1,  $2^{\aleph_{\alpha_n+1}}$  is in  $\mathcal{HCH}_{\omega_1, \omega}$ .

We know that

$$2^{\aleph_\lambda} = (2^{<\aleph_\lambda})^{cf(\aleph_\lambda)},$$

and

$$2^{<\aleph_\lambda} = \sup_n (2^{\aleph_{\alpha_n+1}}).$$

By theorem 3.6,  $2^{<\aleph_\lambda} \in \mathcal{CH}_{\omega_1, \omega}$ , and since  $cf(\aleph_\lambda) = \omega$ , we conclude that  $2^{\aleph_\lambda} \in \mathcal{HCH}_{\omega_1, \omega}$  by theorem 3.2.  $\square$

Combining the last theorem with theorem 4.1 we can conclude:

**Theorem 4.6.** *If  $\aleph_\beta \in \mathcal{CH}_{\omega_1, \omega}$ , then  $2^{\aleph_{\beta+\beta_1}} \in \mathcal{HCH}_{\omega_1, \omega}$ , for all  $0 < \beta_1 < \omega_1$ .*

*Proof.* By induction on  $\beta_1$ . If it is a successor ordinal, use theorem 4.1. If it is a (countable) limit ordinal, then use 4.5.  $\square$

The only case that is not covered by the above theorem is when  $\aleph_\beta$  is the head of a cluster (see definition 1.3) and  $\aleph_\beta \neq \aleph_0$  ( $2^{\aleph_0}$  is easily seen to be in  $\mathcal{HCH}_{\omega_1, \omega}$  by theorem 3.2). Combining with theorem 3.4 we get

**Theorem 4.7.** *If  $\alpha \leq \beta$ ,  $\aleph_\beta \in \mathcal{CH}_{\omega_1, \omega}$ ,  $\alpha_1 < \omega_1$  and  $0 < \beta_1 < \omega_1$ , then*

$$\aleph_{\alpha+\alpha_1}^{\aleph_{\beta+\beta_1}} \in \mathcal{HCH}_{\omega_1, \omega}.$$

*Proof.* By the previous theorem,  $2^{\aleph_{\beta+\beta_1}} \in \mathcal{HCH}_{\omega_1, \omega}$ . Since  $\alpha < \beta$ ,  $\aleph_\alpha^{\aleph_{\beta+\beta_1}} = 2^{\aleph_{\beta+\beta_1}}$  and we conclude by theorem 3.4.  $\square$

Thus, depending on our model of ZFC, we get characterizability of many cardinals that weren't considered before, like  $\aleph_\alpha^{\aleph_\beta}$  for  $\alpha, \beta < \omega_1$  etc.

## 5. OPEN PROBLEMS

There are a few open problems of various difficulties:

- (1) Is there any cardinal that is characterizable by a sentence in  $\mathcal{L}_{\omega_1, \omega}$ , but not characterizable by a Scott sentence? If our model of ZFC satisfies e.g. GCH, then the answer is “No”. So, we should ask if there is any model of ZFC in which this happens. If there is not, then our job becomes a lot easier, since we don’t have to worry about completeness every time. That’s a big step forward. If there is, then it will be very interesting to see one, but I can’t imagine how it would look like. Either way, it seems to be a difficult question.
- (2) Is there any cardinal in  $\mathcal{CH}_{\omega_1, \omega} \setminus \mathcal{HCH}_{\omega_1, \omega}$ , other than  $\aleph_0$ ? Can such a cardinal be a successor? If such a cardinal is a limit cardinal, does it necessarily have cofinality  $\omega$ ? From [4] we know that it is consistent that all cardinals in  $\mathcal{CH}_{\omega_1, \omega} \setminus \mathcal{HCH}_{\omega_1, \omega}$  have cofinality  $\omega$ .

**Conjecture.** For  $\kappa$  a characterizable cardinal,  $\kappa$  is not homogeneously characterizable if and only if it has cofinality  $\omega$ .

- (3) For  $\kappa \in \mathcal{CH}_{\omega_1, \omega}$ , do we get  $2^\kappa \in \mathcal{CH}_{\omega_1, \omega}$ ? This would improve theorem 4.6 and would provide closure under powerset in all cases. If in our model of ZFC,  $\kappa^{\aleph_0} = \kappa$ , then  $\kappa \in \mathcal{HCH}_{\omega_1, \omega}$  and, whence,  $2^\kappa \in \mathcal{HCH}_{\omega_1, \omega}$ . But does it hold for all models?
- (4) If  $\kappa, \lambda$  are both characterizable, do we get  $\lambda^\kappa \in \mathcal{CH}_{\omega_1, \omega}$ ? This would generalize the theorems we prove here. Under GCH, the answer is trivially “Yes”, but is this a theorem in ZFC? By theorem 3.4, we have to consider only the case where  $\kappa$  is the head of a cluster. See also [5] for theorems along this line.
- (5) Are there any closure properties for  $\mathcal{CH}_{\omega_1, \omega}$  (and  $\mathcal{HCH}_{\omega_1, \omega}$ ) besides successor, countable unions, countable products, powerset and powers?

**In memory.** This paper, as long as [4], was written during the academic year 2006-2007, while visiting my thesis advisor professor Greg Hjorth at the University of Melbourne, Australia. Professor Hjorth died in 2011 and this paper is dedicated to his memory.

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